Computational Analysis of Invisibility Cloaking using NURBS

Scott Little

Applications Administrator Analyst, SunWize Technologies

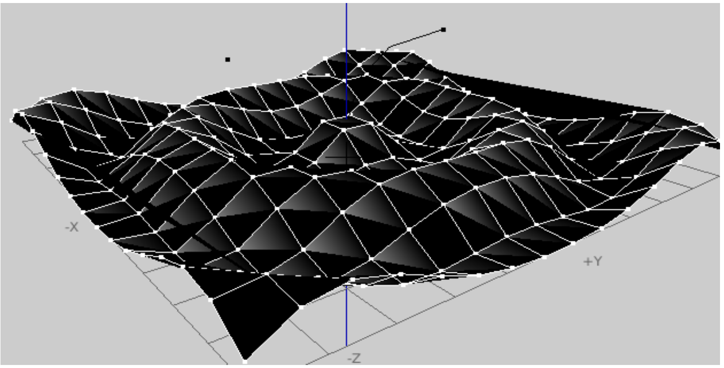
PhD Candidate Applied Computer Science, Northcentral University

Abstract

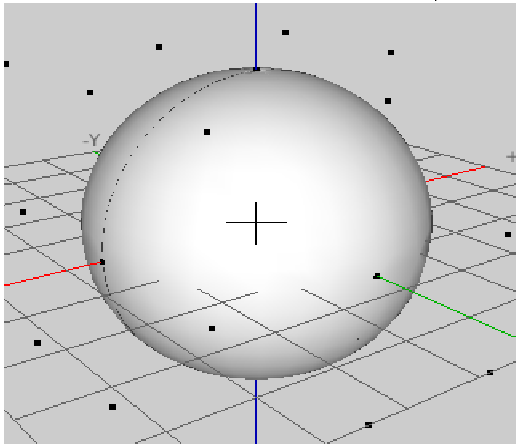
The purpose of this research is to develop a Non-Uniform Rational B-Spline (NURBs) method to accurately measure and graphically represent actions of electromagnetic cloaking specifically defined by the parameters of the Helmholtz Equation.

Using singular transformation optics, the shape representation is that of an electromagnetic wave in topological space. Additional applications include not only invisibility cloaks but other aspects of stealth technology.

Inspiration: SIAM Review article " A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann. Cloaking Devices, Electromagnetic Wormholes, and Transformation Optics . SIAM Review. Doi:10.1137/080716827, March, 2009.



Bezier Triangle Patch Wave Form

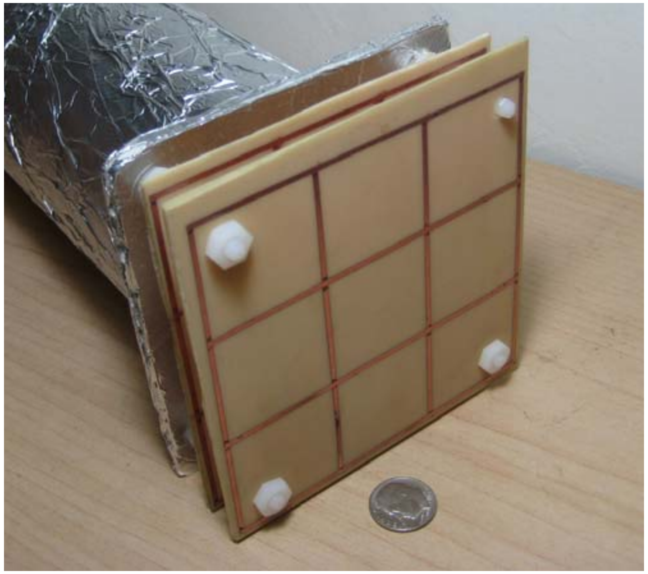


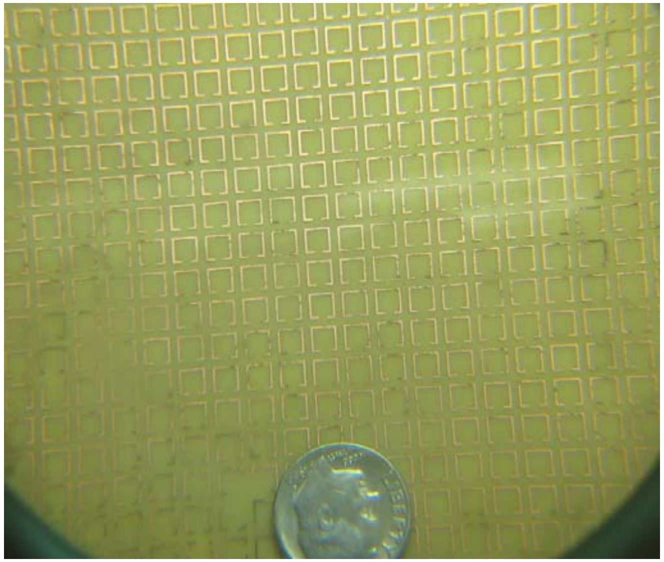
NURBS Circle-K3D

Applications and Future Research

- Invisibility cloaking: light and acoustics
- Solar and other renewable energy-negative refraction
- Wireless transmissions, antennas
- Black holes, gravitational lensing?

- Mad Fellows Labs





Helmholtz Equation

- Why the Helmholtz? Used for both electro-magnetic and acoustic waveforms. The equation is easily translatable, homogeneous and un-homogeneous, includes both scalar and vector forms [2].

- Helmholtz as a second order homogeneous spatial time harmonic PDE with k=2π/λ wave number and solutions using a complex form

$$\frac{\partial^2 \tilde{U}(x)}{\partial x^2} + k^2 \tilde{U}(x) = 0$$

$$\tilde{U}(x) = C_1 e^{j(\omega t - kx)} + C_1 e^{j(\omega t + kx)}$$

NURBS

- A NURB is a non-uniform rational Bezier-spline curve in a piecewise polynomial format [1]. They have been used for years in 3D design and animation. NURBS are created by series of knot vectors defined by control points. The general equation for a NURB in two dimensions is

$$C^{\omega}(u) = \sum_{i=1}^{n} N_{i,p}(u) P_i^{w}$$

- Where P are control point vectors, N is the b-spline function defined by the knot vector, w are the weights.

- Circle example. here is the knot vector for a NURBS circle radius =1. The ratios between the knot values are important and define the shape. The multiplicity is the number of occurrences e.g. 0 is multiplicity 3. $\Xi = \{0,0,0, \pi/2, \pi/2, \pi, \pi, 3\pi/2, 3\pi/2, 2\pi, 2\pi, 2\pi\}$

Isogeometric Analysis

- Isogeometric Analysis was created by the DR. Hughes and the Applied Mathematics and Computational Science team at the University of Texas Austin [1].
- It is an effort to combine CAD and Finite Element design for exact values of mesh and control points.
- NURBS Knot vectors are used to define the mesh.
- Three basic methods: h-method uses knot insertion, p-method uses order of the knots, and k-method is a combination of h and p methods [2].

Transformation Optics

- Transformation Optics uses General Relativity to define curvatures in electromagnetic and acoustic mediums [4].
- The curvature is caused by deflection and bending around an object in optical or acoustic waves.
- Invisibility pioneers Dr. Leonhardt and Dr. Philbin at the University of St. Andrews.

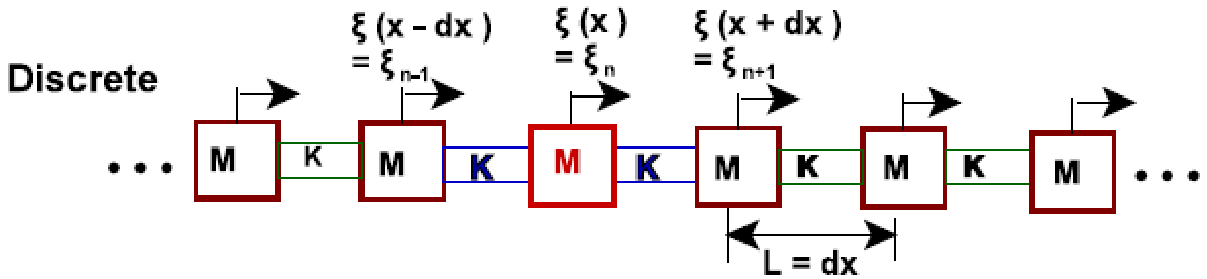
Process Methodology

1. Isogeometric Analysis using Helmholtz Equation
    a. Convert to wave equation, use “mass-spring” method
    b. Solve for “kh” scaling factor, includes wave number
    c. Boundaries are Dirichlet fixed-fixed; discretization
    d. NURB nodes into Excel spreadsheet for graph/simulation

2. Transformation Optics
    a. Convert Helmholtz using change of coordinates
    b. Test with zero curvature scalar verifies transformative cloaking
    c. Solve for “ih” imaginary coordinate value
    d. Transformed values into spreadsheet for transformed simulation, cloaking

Isogeometric Analysis

Discrete notation for mass displacements



M = ρ S dx = ρ S h   and   K = E S /dx = E S /h

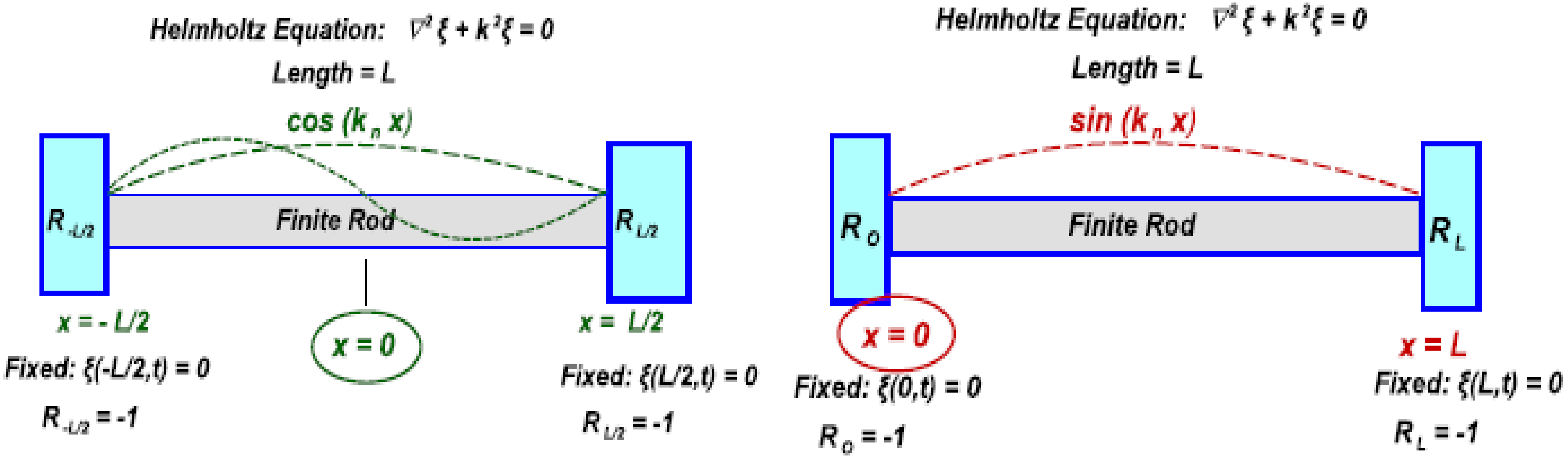
Equation of motion for mass element M that is located at n

$$M \ddot{\xi}_n + K(\xi_n - \xi_{n-1}) + K(\xi_n - \xi_{n+1}) = 0 \quad \leftarrow \textit{Force Free}$$

$$\rho S h \ddot{\xi}_n + ES/h(-\xi_{n-1} + 2\xi_n - \xi_{n+1}) = 0$$

thanks Dr. Alan Stuart, Professor Emeritus Acoustics at Penn State

Isogeometric Analysis



Isogeometric Analysis

- Our version of the Hughes 3 node stencil equations using the "Mass-Spring" method and Simpson's Rule- sorry no "~"

$$\frac{1}{h}\left(\xi_{n-1}-2\xi_n+\xi_{n+1}\right)+\frac{k^2h}{6}\left(\xi_{n-1}+4\xi_n+\xi_{n+1}\right)$$

- Simpson's Rule

$$\int_{x_{n-1}}^{x_{n+1}}\xi(x)dx = \frac{1}{3}\left(\xi_{n-1}+4\xi_n+\xi_{n+1}\right)$$

- Use cosine identity and solution; substitute into equation

$$2\cos(k^h h) = e^{-jhk^h} + e^{jhk^h}$$

$$\xi(x) = A_- e^{-j(\omega t-kx)} + A_+ e^{+j(\omega t+kx)}$$

- And end up with

$$\cos\left(k^h h\right)=\frac{\left(1/h-2hk^2/6\right)}{\left(1/h+hk^2/6\right)} \qquad \left(k^h h\right)=\cos^{-1}\left(\frac{6-2h^2k^2}{6+2h^2k^2}\right)$$

Isogeometric Analysis

- Our version of the Hughes 5 node stencil equations using the "Mass-Spring" method and Boole's Rule

$$\frac{1}{6h}\left(\xi_{n-2}+2\xi_{n-1}-6\xi_{n}+2\xi_{n+1}+\xi_{n+2}\right)+\frac{k^2h}{90}\left(7\xi_{n-2}+32\xi_{n-1}+12\xi_{n}+32\xi_{n+1}+7\xi_{n+2}\right)$$

- Boole's Rule

$$\int_{x_i}^{x_s} f(x)dx = \xi 4h \cong \frac{2}{45}\left(7\xi_{n-2}+32\xi_{n-1}+12\xi_{n}+32\xi_{n+1}+7\xi_{n+2}\right)$$

- And with 2cos identity end up with

$$\cos\left(k^h h\right)=\frac{\left(3h+45h-6\right)}{\left(4+39k^2\right)} \qquad \left(k^h h\right)=\cos^{-1}\left(\frac{\left(3h+45h-6\right)}{\left(4+39k^2\right)}\right)$$

Transformation Optics

- When defining Helmholtz in two dimensions it is convenient to use a complex plane. In addition for optics the wave number $k = \frac{\eta\omega}{c}$ with the refractive index and frequency [3].
- Then the Helmholtz Equation is

$$\nabla^2 \xi(x) + \frac{\eta^2 \omega^2}{c^2} \xi(x) = 0$$

- Using the identity and Chain Rule with complex coordinates:

$$\nabla^2 = \partial_x^2 + \partial_y^2 = \left(\partial_z^2 + \partial_{z*}^2\right) = 4\,\partial_z^2\,\partial_{z*}^2$$

- New Helmholtz Equation with transformed refractive index, using complex plane

$$\left(4\partial_z^2 \partial_{z*}^2 + \frac{\eta^2 \omega^2}{c^2}\right)\psi = 0 \qquad \eta = \left|\frac{dw}{dz}\right| \eta'$$

Transformation Optics

- However this is a scalar, we want a vector equation.
- Begin with vector wave equation

$$\nabla^{j}\nabla_{j}E_{i} - R_{ij}E^{j} - \frac{1}{c^{2}}\frac{\partial E_{i}}{\partial t^{2}} = 0$$

- Set $R_{ij} = 0$ this is test for transformative media, when curvature scalar tensor is zero.
- Now we have the Helmholtz Equation in vector form [3].

$$\nabla^{j}\nabla_{j}E_{i} + \frac{1}{c^{2}}\frac{\partial E_{i}}{\partial t^{2}} = 0$$

- “+” is due negative wave frequency

Transformation Optics

- Now change the coordinates to Partial/Cartesian in Euclidian Space

$$\partial^{j'}\partial_{j'}E_{i'} - \frac{1}{c^2}\frac{\partial E_{i'}}{\partial t^2} = 0$$

- Which enables us to use the old transformed identity and solution [3].

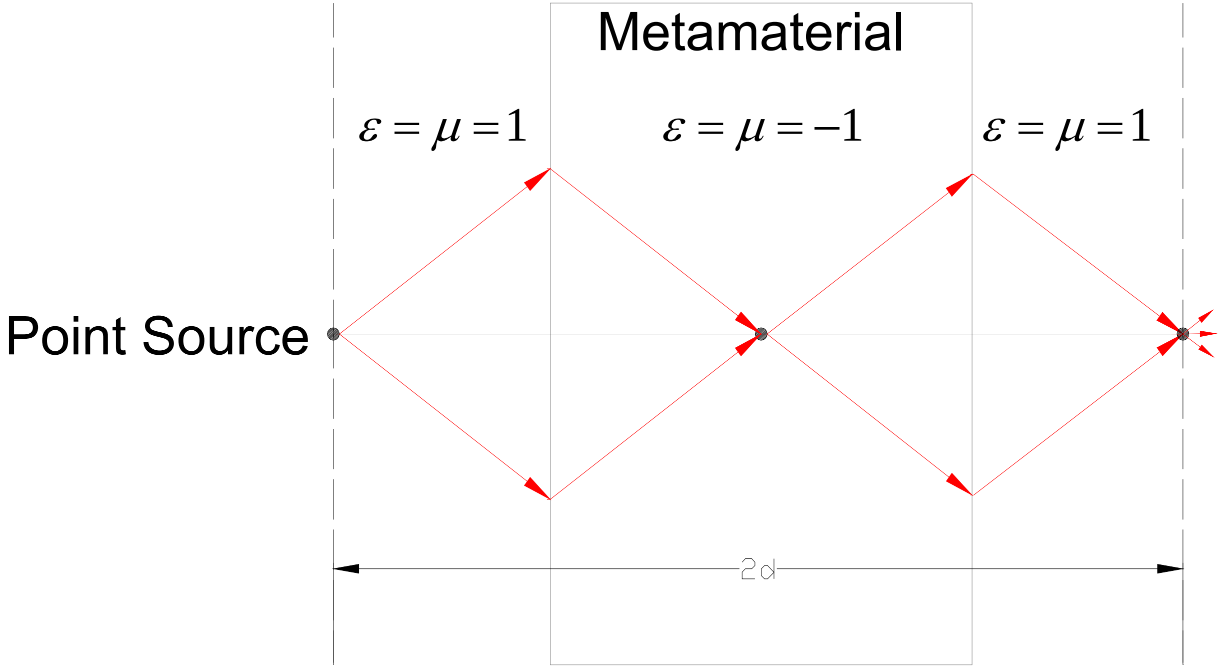
$$E = E_{i'}[2\cos(i^{h'}h)] = E_{i'}[e_{-}^{-jhi^{h'}} + e_{+}^{jhi^{h'}}]$$

- And end up with for the 5 node equation

$$\cos\left(i^{h'}h\right) = \frac{(3h' + 45h' - 6)}{\left(4 + 39k'^2\right)} \qquad \left(i^{h'}h\right) = \cos^{-1}\left(\frac{(3h' + 45h' - 6)}{\left(4 + 39k'^2\right)}\right)$$

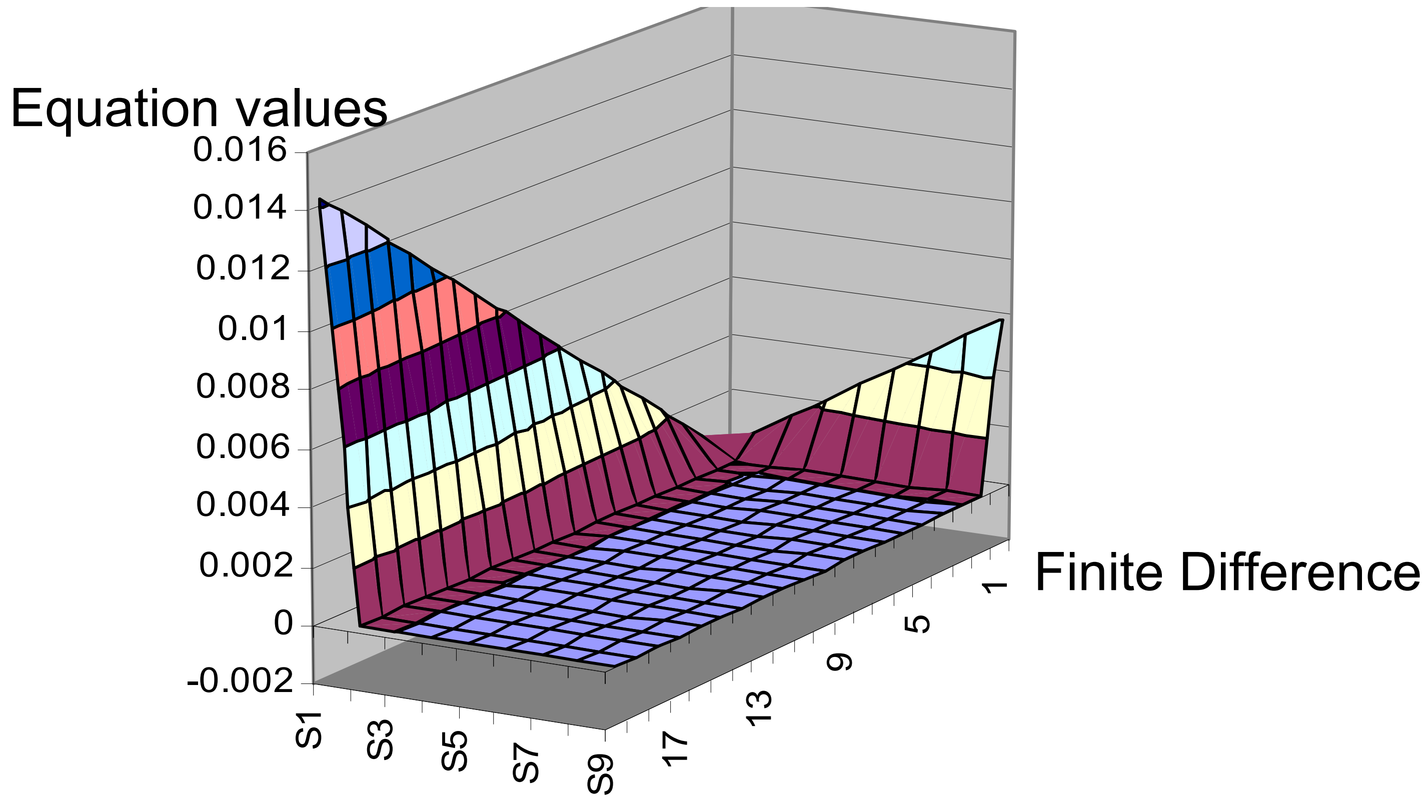
Transformation Optics

- Negative Refraction-Perfect Lensing: refraction index is reversed [3].
- Point in space perfectly imaged on other two.
- Crucial for use with meta-materials i.e. split ring resonators.
- Permittivity and permeability are less than one.



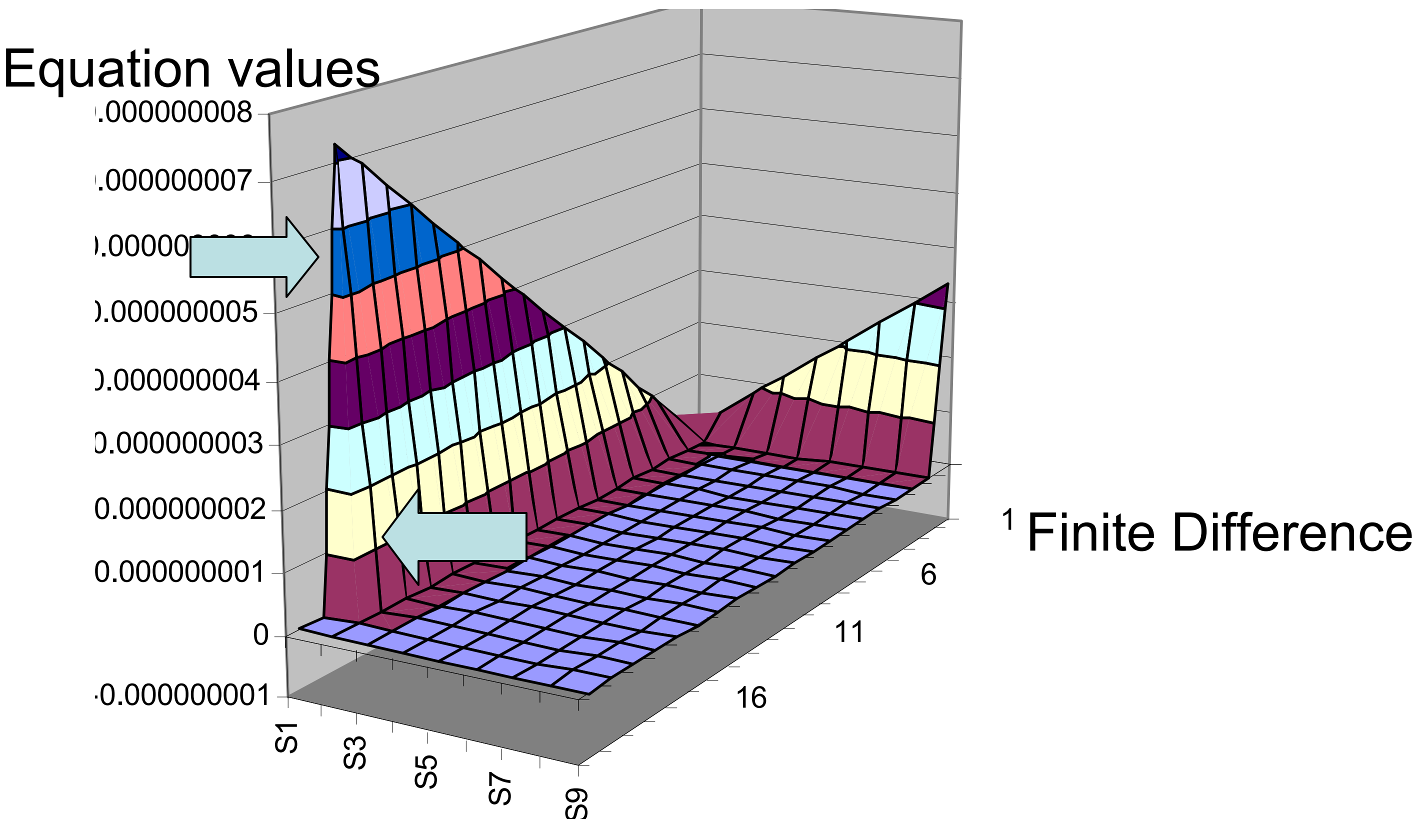
Numerical Simulation 1

- All simulations created in Excel
- Iterative finite difference method for solving 5 node equation
- Graph 20x20 matrix original equation
- Frequency = 5MHz, refractive index =10, “h” knot spacing= 20



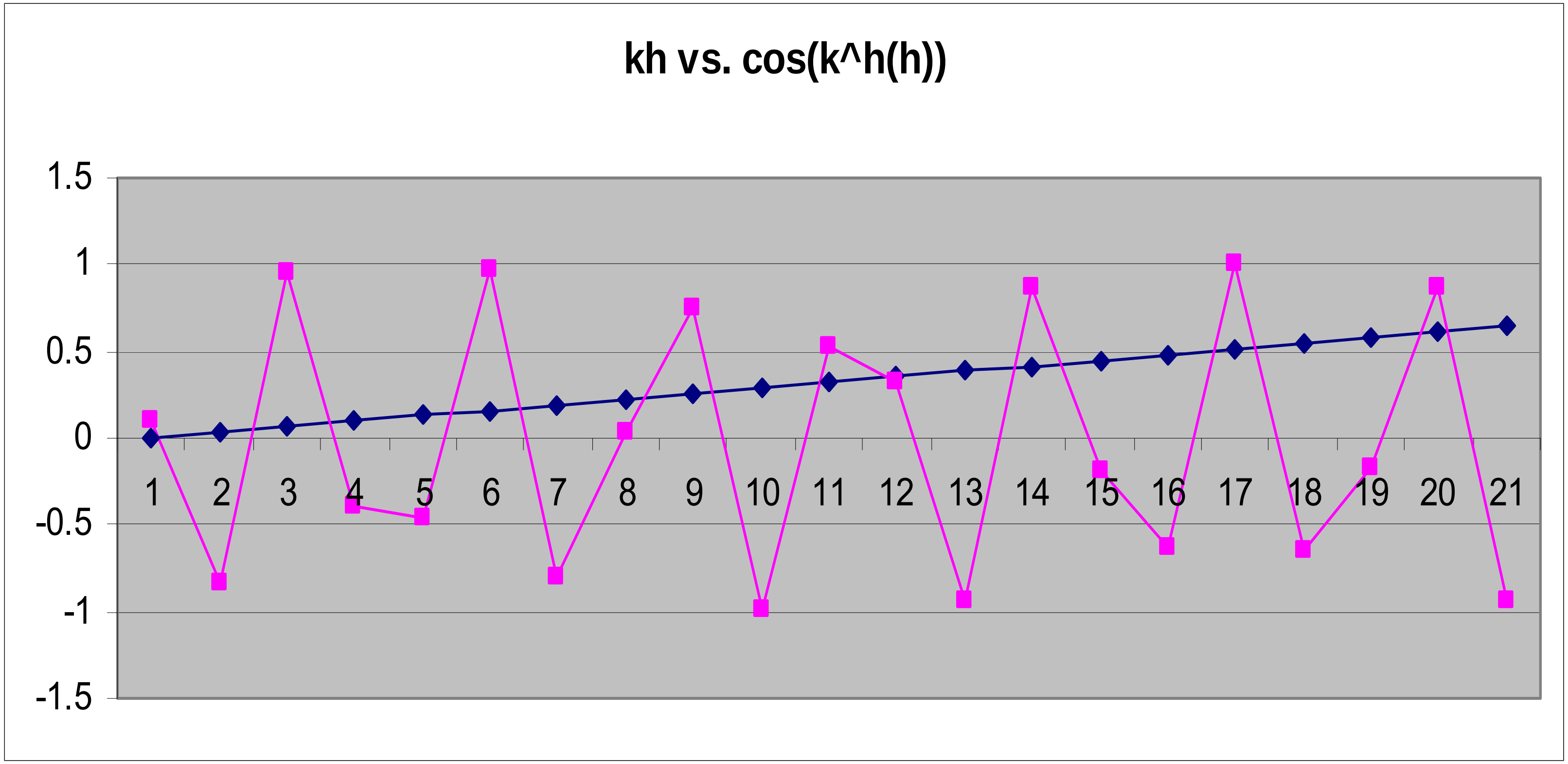
Numerical Simulation 2

- Iterative finite difference method for solving 5 node equation
- Graph 20x20 matrix transformed equation; cloaked area is “tent”
- Negative values below 1 MHz



Numerical Simulation 3

- Graph kh vs. cos(k^h(h))
- Cos fluctuates around kh; cos is magenta
- y axis are computed values, x axis are "h" entered values



References

- [1] T. Hughes, J. Cottrell and Y. Bazilevs. Isogeometric Analysis: CAD, Finite Elements, NURBS, Exact Geometry and Mesh Refinement . Comput. Methods Appl. Mech. Engrg. doi:10.1016/j.cma.2004.10.008, 20 October 2004.

- [2] T. Hughes, A. Reali and G. Sangalli. The Duality and Unified Analysis of Discrete Approximations in Structural Dynamics and Wave Propagation: Comparison of p-method Finite Elements with k-method NURBS Einstein-Dirac Equation on Riemannian Spin Manifolds. Elsevier Science, 10 October 2007.

- [3] U. Leonhardt and T. Philbin. Geometry and Light: The Science of Invisibility. Dover Publications, N.Y., 2010.

- [4] U. Leonhardt and T. Philbin. Transformation Optics and the Geometry of Light.arXiv:0805.4778v2 [physics.optics] ,7 Jun 2008.